

Vortex lattice solutions to the Gross-Pitaevskii equation with spin-orbit coupling in optical lattices

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Effective spin-orbit coupling can be created in cold atom systems using atom-light interaction. We study the BECs in an optical lattice using the Gross-Pitaevskii equation with spin-orbit coupling. Bloch states for the linear equation are numerically obtained, and compared with stationary solutions to the Gross-Pitaevskii equation with nonlinear terms. Various vortex lattice states are found when the spin-orbit coupling is strong.

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Recently, Bose-Einstein condensates (BECs) with effective spin-orbit coupling were created in cold atom systems using atom-light interaction [1]. The spin-orbit-coupled BECs are actively studied theoretically [2]. Wang et al. found that the mean-field ground state has two different phases: plane-wave and stripe phases depending on the nonlinear interactions [3]. Half vortex states were found in a spin-orbit coupled BECs confined in a harmonic potential [4, 5]. Exotic spin textures were predicted in Bose-Hubbard models corresponding to spin-orbit coupled BECs in the Mott-insulator phase [6, 7].

The Gross-Pitaevskii (GP) equation is a mean-field approximation for the BECs with the spin-orbit coupling. There are some studies for the GP equation with spin-orbit coupling in optical lattices [8, 9]. In this paper, we study vortex lattice solutions to the GP equation in a square type optical lattice. The model equation is expressed as

$$\begin{aligned} i\frac{\partial\psi_+}{\partial t} &= -\frac{1}{2}\nabla^2\psi_+ + (g|\psi_+|^2 + \gamma|\psi_-|^2)\psi_+ - \epsilon\{\cos(2\pi x) + \cos(2\pi y)\}\psi_+ + \lambda\left(\frac{\partial\psi_-}{\partial x} - i\frac{\partial\psi_-}{\partial y}\right), \\ i\frac{\partial\psi_-}{\partial t} &= -\frac{1}{2}\nabla^2\psi_- + (g|\psi_-|^2 + \gamma|\psi_+|^2)\psi_- - \epsilon\{\cos(2\pi x) + \cos(2\pi y)\}\psi_- + \lambda\left(-\frac{\partial\psi_+}{\partial x} - i\frac{\partial\psi_+}{\partial y}\right), \end{aligned} \quad (1)$$

where $\psi = (\psi_+, \psi_-)$ denotes the wave function of the spinor BECs, ϵ is the strength of the optical lattice, g and γ express the strengths of interactions respectively between the same and the different kinds of atoms, and λ denotes the strength of the Rashba spin-orbit coupling. We have assumed that the wavelength of the optical lattice is 1.

If g and γ are zero, Eq. (1) becomes linear equations with spatially-periodic potential. The Bloch states are stationary solutions to the linear equations, which are expressed as

$$\psi_+(x, y, t) = \phi_+(x, y) \exp(ik_x x + ik_y y - i\mu t), \quad \psi_-(x, y, t) = \phi_-(x, y) \exp(ik_x x + ik_y y - i\mu t), \quad (2)$$

where ϕ_+ and ϕ_- are periodic functions of wavelength 1. Therefore, ϕ_+ and ϕ_- satisfy

$$\begin{aligned} \mu\phi_+ &= -\frac{1}{2}\nabla^2\phi_+ + \frac{1}{2}(k_x^2 + k_y^2)\phi_+ - ik_x\frac{\partial\phi_+}{\partial x} - ik_y\frac{\partial\phi_+}{\partial y} - \epsilon\{\cos(2\pi x) + \cos(2\pi y)\}\phi_+ \\ &\quad + \lambda\left(\frac{\partial\phi_-}{\partial x} - i\frac{\partial\phi_-}{\partial y} + ik_x\phi_- + ik_y\phi_-\right), \\ \mu\phi_- &= -\frac{1}{2}\nabla^2\phi_- + \frac{1}{2}(k_x^2 + k_y^2)\phi_- - ik_x\frac{\partial\phi_-}{\partial x} - ik_y\frac{\partial\phi_-}{\partial y} - \epsilon\{\cos(2\pi x) + \cos(2\pi y)\}\phi_- \\ &\quad + \lambda\left(-\frac{\partial\phi_+}{\partial x} - i\frac{\partial\phi_+}{\partial y} - ik_x\phi_+ + ik_y\phi_+\right). \end{aligned} \quad (3)$$

The eigenvalue μ and the eigen function ϕ_+ and ϕ_- can be numerically obtained from the stationary solution of the

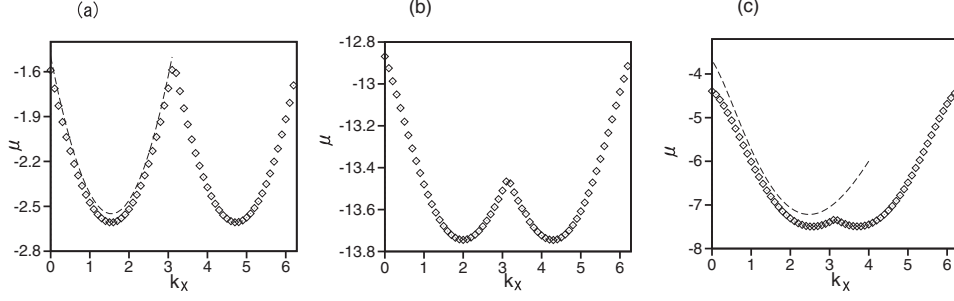


FIG. 1: Eigenvalue μ vs. k_x for (a) $\lambda = \pi/2, k_y = 0$, (b) $\lambda = 3\pi/2, k_y = 0$, and (c) $\lambda = \pi, k_y = k_x$. Dashed curve in Fig.1(a) is plotted using Eq. (8) and the dashed curve in Fig. 1(c) is obtained using Eq. (6).

linear equation [10]:

$$\begin{aligned}
 \frac{\partial \phi_+}{\partial t} &= \frac{1}{2} \nabla^2 \phi_+ - \frac{1}{2} (k_x^2 + k_y^2) \phi_+ + i k_x \frac{\partial \phi_+}{\partial x} + i k_y \frac{\partial \phi_+}{\partial y} + \epsilon \{ \cos(2\pi x) + \cos(2\pi y) \} \phi_+ \\
 &\quad - \lambda \left(\frac{\partial \phi_-}{\partial x} - i \frac{\partial \phi_-}{\partial y} + i k_x \phi_- + k_y \phi_- \right) + \mu \phi_+, \\
 \frac{\partial \phi_-}{\partial t} &= \frac{1}{2} \nabla^2 \phi_- - \frac{1}{2} (k_x^2 + k_y^2) \phi_- + i k_x \frac{\partial \phi_-}{\partial x} + i k_y \frac{\partial \phi_-}{\partial y} + \epsilon \{ \cos(2\pi x) + \cos(2\pi y) \} \phi_- \\
 &\quad - \lambda \left(-\frac{\partial \phi_+}{\partial x} - i \frac{\partial \phi_+}{\partial y} - i k_x \phi_+ + k_y \phi_+ \right) + \mu \phi_-, \\
 \frac{d\mu}{dt} &= \alpha (N_0 - N),
 \end{aligned} \tag{4}$$

where $\alpha > 0$ is a parameter and fixed to be 5 in our numerical simulation. $N = \int_0^1 \int_0^1 (|\phi_+|^2 + |\phi_-|^2) dx dy$ is the total norm, and N_0 is fixed to be 1 by the normalization condition. The time evolution of the dissipative equation (4) leads to a stationary state and the total norm N approaches $N_0 = 1$. The eigenvalue μ in Eq. (3) is obtained as μ in Eq. (4) at the stationary state. In this numerical method, the ground state for fixed values of k_x and k_y is obtained at the stationary state, starting from most initial conditions, because the total energy decreases in the time evolution of Eq. (4). Excited states are obtained by removing the ground state by the method of orthogonalization. Figure 1(a) shows $\mu(k_x)$ as a function of k_x for $k_y = 0, \epsilon = 5$, and $\lambda = \pi/2$. $\mu(k_x)$ is a periodic function of k_x with period 2π . There are peaks near $k_x = 0, \pi$ and 2π and minima at $k_x \sim \pi/2$ and $3\pi/2$. The peak point at $k_x = \pi$ is a cusp point, where two $\mu(k_x)$ curves corresponding to the ground state and the excited state cross, although the branch of the excited state is not shown. For $\lambda = 0$, $\mu(k_x)$ increases monotonously as k_x increases from 0 and reaches the maximum at the edge of the Brillouin zone at $k_x = \pi$. If there is no optical lattice, i.e., $\epsilon = 0$, $\mu(k)$ takes a minimum at $k = \lambda$ where $k = \sqrt{k_x^2 + k_y^2}$ [2, 3]. The minimum point for $\lambda = \pi/2$ locates near $k_x = \lambda$, and the peak corresponds to the edge of the Brillouin zone. Figure 1(b) shows $\mu(k_x)$ at $\lambda = (3/2)\pi$. There are a large peak at $k_x = 0$ and 2π and a small peak at $k_x = \pi$ and minima at $k_x = 2$ and $k_x = 4.3$. Figure 1(c) shows $\mu(k)$ as a function of k_x for $k_x = k_y, \lambda = \pi, \epsilon = 5$. There is a large peak at $k_x = 0$ and 2π , a small peak at $k_x = \pi$, and minima at $k_x \sim 2.5$ and 3.8 . The wavenumber $k_x \sim 2.5$ is close to $\lambda/\sqrt{2} \sim 2.22$ by the simplest approximation $k = \lambda$ but slightly deviated. The approximation $k_{\min} = \lambda$ for the minimum point of $\mu(k)$ becomes worse for large λ . The small peaks correspond to the edge of the Brillouin zone.

Figure 2(a) shows $|\phi_+(x, y)|$ and $|\phi_-(x, y)|$ as a function of y in the section $x = 0$ at $k_x = 3\pi/2$ for $\lambda = 3\pi/2$. The modulus $|\phi_+|$ and $|\phi_-|$ take maximum at different positions. The minimum value is almost zero, which implies the existence of vortices. Figure 2(b) shows a contour plot of $|\phi_+|$ for the same parameter. The locations of vortices for ϕ_+ can be calculated from the phase distribution $\theta_+(x, y) = \sin^{-1}(\text{Im } \phi_+(x, y)/|\phi_+(x, y)|)$. There exist a vortex at a point, if the integral of the phase gradient along an anticlockwise path encircling the point is a nontrivial multiple of 2π . We have counted the path integral by discretizing the (x, y) space with $\Delta x = 1/64$. Figure 2(c) shows positions of vortices of vorticity ± 1 with square and \times marks. The vortex cores locate near $(0, -0.28)$ and $(0, -0.48)$ for ϕ_+ . In generic cases, there is a vortex of vorticity 1 or -1 at a position satisfying $|\phi_+| = 0$, where a line of $\text{Re } \phi_+ = 0$ intersects with a line of $\text{Im } \phi_+ = 0$ [11]. We do not show explicitly the positions of vortices later in Fig. 3 and Fig. 4, however, we have checked the existence of vortices of vorticity 1 or -1 at positions satisfying $|\phi_{\pm}| = 0$ by calculating

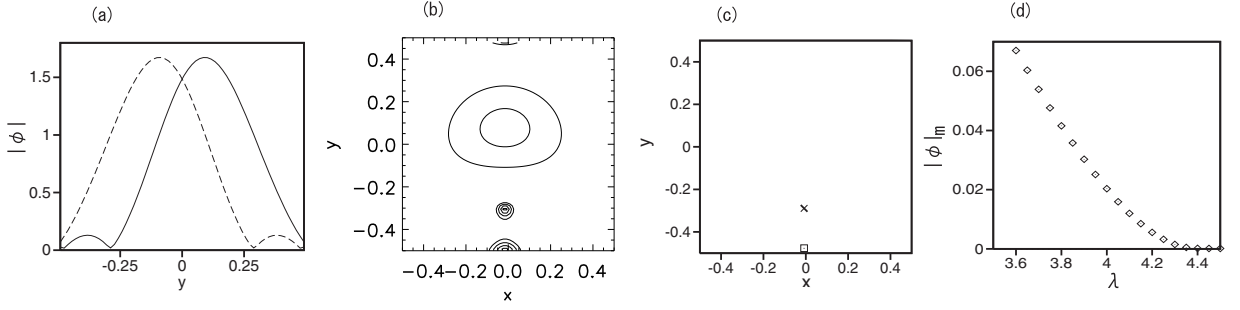


FIG. 2: (a) $|\phi_+|$ (solid curve) and $|\phi_-|$ (dashed curve) along the line $x = 0$ for $\lambda = 3\pi/2$, and $k_x = 3\pi/2$. (b) Contour plot of $|\phi_+|$. (c) Square shows a vortex with vorticity 1 and \times shows a vortex with vorticity -1. (d) Minimum values of $|\phi_+|$ as a function of λ for $k_x = \lambda$ and $k_y = 0$.

the phase distribution. Figure 2(d) shows the minimum value of $|\phi_+|$ as a function of λ for $k_x = \lambda, k_y = 0$ at $\epsilon = 5$. The minimum value becomes zero and a vortex-antivortex pair appears for $\lambda > \lambda_c \sim 4.2$.

Because ϕ_{\pm} are periodic functions with wavelength 1, ϕ_{\pm} can be expressed as the simplest approximation:

$$\begin{aligned}\phi_+ &= C_{0+} + C_{1+}e^{2\pi ix} + C_{2+}e^{-2\pi ix} + C_{3+}e^{2\pi iy} + C_{4+}e^{-2\pi iy}, \\ \phi_- &= C_{0-} + C_{1-}e^{2\pi ix} + C_{2-}e^{-2\pi ix} + C_{3-}e^{2\pi iy} + C_{4-}e^{-2\pi iy}.\end{aligned}\quad (5)$$

Substitution of this ansatz into Eq. (3) yields

$$\begin{aligned}\mu C_{0\pm} &= (k_x^2 + k_y^2)C_{0\pm}/2 - (\epsilon/2)(C_{1\pm} + C_{2\pm} + C_{3\pm} + C_{4\pm}) + \lambda(\pm ik_x + k_y)C_{0\mp}, \\ \mu C_{1\pm} &= \{(k_x + 2\pi)^2 + k_y^2\}C_{1\pm}/2 - (\epsilon/2)C_{0\pm} + \lambda\{\pm i(k_x + 2\pi) + k_y\}C_{1\mp}, \\ \mu C_{2\pm} &= \{(k_x - 2\pi)^2 + k_y^2\}C_{2\pm}/2 - (\epsilon/2)C_{0\pm} + \lambda\{\pm i(k_x - 2\pi) + k_y\}C_{2\mp}, \\ \mu C_{3\pm} &= \{k_x^2 + (k_y + 2\pi)^2\}C_{3\pm}/2 - (\epsilon/2)C_{0\pm} + \lambda\{\pm ik_x + (k_y + 2\pi)\}C_{3\mp}, \\ \mu C_{4\pm} &= \{k_x^2 + (k_y - 2\pi)^2\}C_{4\pm}/2 - (\epsilon/2)C_{0\pm} + \lambda\{\pm ik_x + (k_y - 2\pi)\}C_{4\mp}.\end{aligned}\quad (6)$$

For $k_y = 0$, $C_{0-} = iC_{0+}$, $C_{1-} = iC_{1+}$, $C_{2-} = iC_{2+}$ are satisfied, and then

$$\begin{aligned}C_{1+} &= \frac{-(\epsilon/2)C_{0+}}{\mu - (k_x + 2\pi)^2/2 + \lambda(k_x + 2\pi)}, \\ C_{2+} &= \frac{-(\epsilon/2)C_{0+}}{\mu - (k_x - 2\pi)^2/2 + \lambda(k_x - 2\pi)}, \\ C_{3+} &= \frac{[-(\epsilon/2)\{\mu - (k_x^2 + 4\pi^2)/2\} + (\epsilon/2)\lambda(k_x - 2\pi i)]C_{0+}}{\{\mu - (k_x^2 + 4\pi^2)/2\}^2 - \lambda^2(k_x^2 + 4\pi^2)}, \\ C_{4+} &= \frac{[-(\epsilon/2)\{\mu - (k_x^2 + 4\pi^2)/2\} + (\epsilon/2)\lambda(k_x + 2\pi i)]C_{0+}}{\{\mu - (k_x^2 + 4\pi^2)/2\}^2 - \lambda^2(k_x^2 + 4\pi^2)},\end{aligned}\quad (7)$$

where μ is given by a solution of the equation

$$\begin{aligned}\mu &= \frac{k_x^2}{2} - \lambda k_x + \frac{\epsilon^2/4}{\mu - (k_x + 2\pi)^2/2 + \lambda(k_x + 2\pi)} + \frac{\epsilon^2/4}{\mu - (k_x - 2\pi)^2/2 + \lambda(k_x - 2\pi)} \\ &\quad + \frac{\epsilon^2}{4} \frac{2\mu - (k_x^2 + 4\pi^2) - 2\lambda k_x}{\{\mu - (k_x^2 + 4\pi^2)/2\}^2 - \lambda^2(k_x^2 + 4\pi^2)}.\end{aligned}\quad (8)$$

Furthermore, $C_{3-} = iC_{3+}^*$, $C_{4-} = iC_{4+}^*$ are satisfied. Here, $*$ denotes the complex conjugate. The dashed curve in Fig. 1(a) denotes $\mu(k_x)$ by Eq. (8) at $\lambda = \pi/2$. The approximation is good for $\lambda = \pi/2$ but is not so good for large λ , because the higher harmonics is necessary for the expansion in Eq. (5). We can assume that C_{0+} , C_{1+} and C_{2+} are real numbers and $C_{4+} = C_{3+}^* = \text{Re}C_{3+} - i\text{Im}C_{3+}$. Then, ϕ_+ and ϕ_- are expressed as

$$\begin{aligned}\phi_+ &= C_{0+} + (C_{1+} + C_{2+})\cos(2\pi x) + i(C_{1+} - C_{2+})\sin(2\pi x) + 2\text{Re}C_{3+}\cos(2\pi y) - 2\text{Im}C_{3+}\sin(2\pi y), \\ \phi_- &= iC_{0+} + i(C_{1+} + C_{2+})\cos(2\pi x) - (C_{1+} - C_{2+})\sin(2\pi x) + 2i\text{Im}C_{3+}\sin(2\pi y) + 2i\text{Re}C_{3+}\cos(2\pi y).\end{aligned}\quad (9)$$

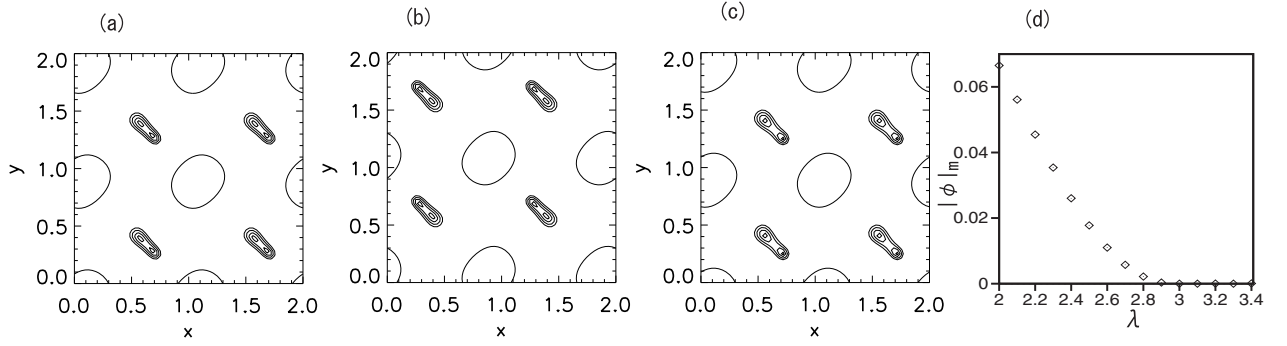


FIG. 3: (a) Contour plot of $|\psi_+|$ for $\lambda = \pi, g = 1, \gamma = 0.5$ and $L = 8$. $k_x = k_y$ are evaluated as $3\pi/4$. (b) Contour plot of $|\psi_-|$. (c) Contour plot of $|\phi_+|$ to the linear equation Eq. (3) for $\lambda = \pi, k_x = k_y = 3\pi/4$. (d) Minimum values of $|\phi_+|$ as a function of λ for $k_x = k_y = \lambda/\sqrt{2}$.

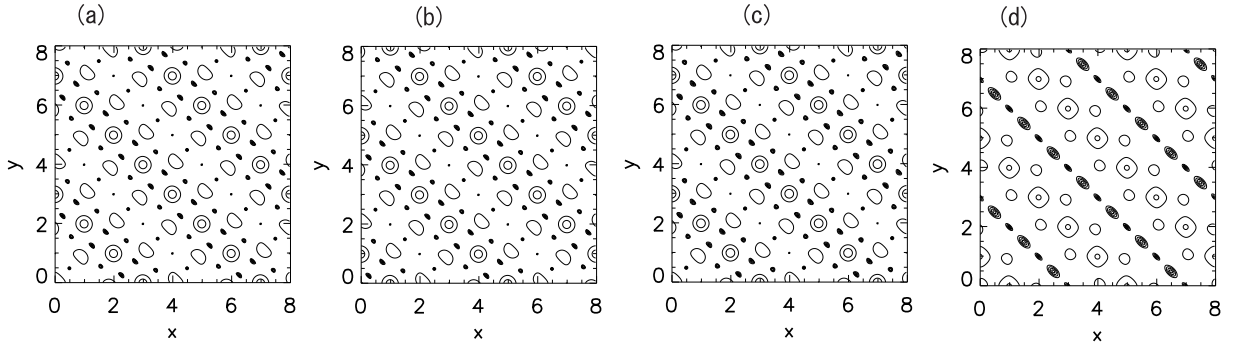


FIG. 4: (a) Contour plot of $|\psi_+|$ for $\lambda = \pi, g = 1, \gamma = 2$ and $L = 8$. $k_x = k_y$ are evaluated as $3\pi/4$. (b) Contour plot of $|\psi_-|$. (c) Contour plot of the superposition of $|(\phi_{++} + \phi_{+-})/\sqrt{2}|$ to the linear equation Eq. (3) for $\lambda = \pi$ and $k_x = k_y = \pm 3\pi/4$. (d) Contour plot of the superposition of $|(\phi_{++} + \phi_{+-})/\sqrt{2}|$ to the linear equation Eq. (3) for $\lambda = 1$ and $k_x = k_y = \pm \pi/4$.

$\text{Im } \phi_+ = 0$ and $\text{Re } \phi_- = 0$ are satisfied on the line $x = 0$. When λ is small, the minimum values of $\text{Re } \phi_+$ and $\text{Im } \phi_-$ are positive and there are no vortices. When λ is increased the minimum values decrease and reach 0, and then a vortex pair is created. A vortex core of ϕ_+ is located at a point on the line $x = 0$ where $\text{Re } \phi_+(0, y) = 0$ is satisfied, and similarly a vortex core of ϕ_- is located at a point on the line $x = 0$ where $\text{Im } \phi_-(0, y) = 0$ is satisfied.

Even for g and γ is not zero, the Bloch state is a good approximation for the stationary state for $\gamma < g$. We have performed numerical simulation of Eq. (1) by the imaginary time evolution method similar to Eq. (4) and found stationary solutions. The system size is $L_x \times L_y = L \times L$ and the total norm $N = \int_0^L \int_0^L (|\psi_+|^2 + |\psi_-|^2) dx dy$ is set to be L^2 in this paper. Periodic boundary conditions are imposed. The potential is shifted as $U = -\epsilon[\cos\{2\pi(x - 1/2)\} + \cos\{2\pi(y - 1/2)\}]$ by $(1/2, 1/2)$ to confine the wave pattern in the range of $[0, L] \times [0, L]$.

Figure 3(a) and (b) show contour plots of $|\psi_+|$ and $|\psi_-|$ at $g = 1, \gamma = 0.5, L = 8, \lambda = \pi$, and $\epsilon = 5$. The contour plot is drawn in the region $[0, 2] \times [0, 2]$, and the contour lines are drawn for $|\psi_{\pm}| = 0.025, 0.05, 0.075, 0.1, 1$ and 1.5 . Vortex pairs exist in each cell of size 1 for this parameter, and a vortex lattice is constructed as a whole. Vortex lattices were experimentally found first in rotating BECs [12] and recently in BECs under synthetic magnetic fields by atom-light interaction [13]. In our model equation, vortices are spontaneously created by the spin-orbit coupling. The wavevector (k_x, k_y) is evaluated at $(3\pi/4, 3\pi/4)$. Positions of vortex cores for ψ_+ and ψ_- are mutually deviated. Figure 3(c) shows a contour plot of $|\phi_+|$ for the linear equation corresponding to $g = 0, \gamma = 0$ for $k_x = k_y = 3\pi/4$ at $\lambda = \pi$ and $\epsilon = 5$. The eigenvalue μ takes a minimum at $(k_x, k_y) = (3\pi/4, 3\pi/4)$ in the finite size system of $L = 8$, where k_x (k_y) takes a discrete value $2\pi n_x/L$ ($2\pi n_y/L$) with integer n_x (n_y). The contour plot is almost the same as Fig. 3(a). It means that the Bloch wave is a good approximation for the solution to the GP equation. Figure 3(d) shows the minimum values of $|\phi_+|$ for the linear equation as a function of λ for $k_x = k_y = \lambda/\sqrt{2}$ at $\epsilon = 5$. The minimum value becomes zero and vortices appear for $\lambda > 2.9$. It is related to the existence of vortices at $\lambda = \pi$.

Stripe wave states are expected to appear for $\gamma > g$. The superposition of Bloch waves of (k_x, k_y) and $(-k_x, -k_y)$ is a simple approximation for $\gamma > g$. Figure 4(a) and (b) show contour plots of $|\psi_+|$ and $|\psi_-|$ at $g = 1, \gamma = 2, L = 8$,

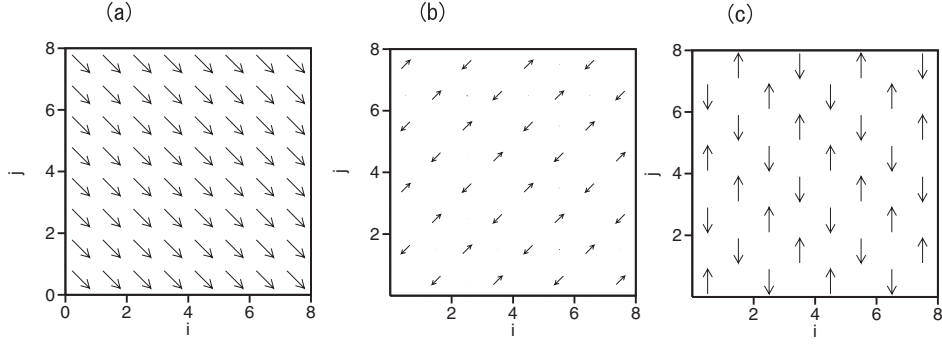


FIG. 5: (a) Spin configuration of $(s_x(i, j), s_y(i, j))$ at $\lambda = \pi, g = 1, \gamma = 0.5$ and $L = 8$. (b) Spin configuration of $(s_x(i, j), s_y(i, j))$ at $\lambda = \pi, g = 1, \gamma = 2$ and $L = 8$. (c) Spin configuration of $s_z(i, j)$ at $\lambda = \pi, g = 1, \gamma = 2$ and $L = 8$.

and $\lambda = \pi$. The wavevector is evaluated as $(k_x, k_y) = (3\pi/4, 3\pi/4)$ in this case, too. The contour lines are drawn for $|\psi_{\pm}| = 0.025, 0.05, 0.075, 0.1, 1$ and 1.5 . Vortex cores exist in dark pointed regions. The vortex lattice structure is rather complicated. The circular contour lines correspond to peak regions of $|\psi_{\pm}|$. The peak regions stand in a line in the direction of angle $-\pi/4$ and the peak lines for ψ_+ and ψ_- alternates in the diagonal direction of angle $\pi/4$. Figure 4(c) shows a contour plot of a linear combination $|(\phi_{++} + \phi_{+-})/\sqrt{2}|$ of two Bloch waves ϕ_{++} and ϕ_{+-} with $(k_x, k_y) = (3\pi/4, 3\pi/4)$ and $(-3\pi/4, -3\pi/4)$ for the $+$ component at $\lambda = \pi$. The superposition of the Bloch waves is a good approximation for the stationary solution to the GP equation. The superposition of two Bloch waves with opposite wavevectors generates a standing wave. For plane waves, the amplitude becomes zero at the nodal lines. The nodal lines are perturbed by the optical lattice and vortices are generated. A vortex lattice structure therefore appears even for small λ in case of $\gamma > g$. Figure 4(d) shows a vortex lattice pattern with $k_x = k_y = \pi/4$ at $\lambda = 1$ and $\epsilon = 5$. For large λ , a vortex pair is created in a single Bloch wave and the superposition of the two Bloch waves make the vortex lattice structure more complicated as shown in Fig. 4(c).

The complicated patterns might be simplified, if a spin representation is used, which was discussed in the Bose-Hubbard model [6, 7]. The whole system is divided into cell regions of $[i - 1, i] \times [j - 1, j]$. The spin variables $s_x(i, j)$, $s_y(i, j)$ and $s_z(i, j)$ are defined for each cell labeled by (i, j) as

$$\begin{aligned} s_x(i, j) &= \int_{i-1}^i \int_{j-1}^j \psi^\dagger \sigma_x \psi dx dy = \int_{i-1}^i \int_{j-1}^j (\psi_+^* \psi_- + \psi_-^* \psi_+) dx dy, \\ s_y(i, j) &= \int_{i-1}^i \int_{j-1}^j \psi^\dagger \sigma_y \psi dx dy = \int_{i-1}^i \int_{j-1}^j (-i\psi_+^* \psi_- + i\psi_-^* \psi_+) dx dy, \\ s_z(i, j) &= \int_{i-1}^i \int_{j-1}^j \psi^\dagger \sigma_z \psi dx dy = \int_{i-1}^i \int_{j-1}^j (|\psi_+|^2 - |\psi_-|^2) dx dy, \end{aligned} \quad (10)$$

where σ_x, σ_y and σ_z are the Pauli matrix, and † denotes the complex conjugate transpose. Figure 5(a) shows (s_x, s_y) corresponding to the pattern in Figs. 3(a) and (b) for $g = 1, \gamma = 0.5, \lambda = \pi$ and $\epsilon = 5$. The vector $(s_x(i, j), s_y(i, j))$ is expressed as an arrow on each lattice point at $(i - 1/2, j - 1/2)$. The pattern is interpreted as a ferromagnetic state in the (x, y) plane in this spin representation. The spin s_z is zero for this pattern. Figures 5(b) and (c) show spin configurations respectively for (s_x, s_y) and s_z for the pattern at $g = 1$ and $\gamma = 2$ shown in Figs. 4(a) and (b). The spin configuration is also rather complicated. The wavelength of the spin configuration is 4 both in the i and j directions. An anti-ferromagnetic order is seen in the diagonal direction of angle $\pi/4$ and a ferromagnetic order appears in its orthogonal direction of angle $-\pi/4$ both for the (s_x, s_y) and s_z patterns. The (s_x, s_y) component appears at the sites where the s_z component vanishes, and the s_z component appears at the sites where the (s_x, s_y) component vanishes.

To summarize, we have studied the Gross-Pitaevskii equation with spin-orbit coupling in an optical lattice. We have found that a vortex lattice structure appears for large λ in case of $\gamma < g$. A vortex lattice structure appears even for small λ in case of $\gamma > g$, because the nodal lines in the stripe wave pattern are perturbed by the optical lattice. We have found a complicated spin configuration in a case of $\gamma > g$. The complicated patterns can be qualitatively understood by the corresponding Bloch waves. The Bloch waves are further approximated by a Fourier series expansion with five modes to understand the formation of the vortices. We have obtained various spin configurations by changing the

parameter λ . The detailed phase diagrams by changing various parameters are under study.

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